

# Index-modules and applications.

Stéphane VIGUIÉ \*

January 13, 2013

**Abstract.** Let  $K$  be a commutative field,  $A \subseteq K$  be a Dedekind ring and  $V$  be a  $K$ -vector space. For any pair of  $A$ -lattices  $R \neq 0$  and  $S$  of  $V$ , we define an  $A$ -submodule  $[R : S]'_A$  of  $K$ , their  $A$ -index-module. Once the basic properties of these modules are stated, we show that this notion can be used to recover more usual ones: the group-index, the relative invariant, the Fitting ideal of  $R/S$  when  $S \subseteq R$ , and the generalized index of Sinnott. As an example, we consider the following situation. Let  $F/k$  be a finite abelian extension of global function fields, with Galois group  $G$ , and degree  $g$ . Let  $\infty$  be a place of  $k$  which splits completely in  $F/k$ . Let  $\mathcal{O}_F$  be the ring of functions of  $F$ , which are regular outside the places of  $F$  sitting over  $\infty$ . Then one may use Stark units to define a subgroup  $\mathcal{E}_F$  of  $\mathcal{O}_F^\times$ , the group of units of  $\mathcal{O}_F$ . We use the notion of index-module to prove that for every nontrivial irreducible rational character  $\psi$  of  $G$ , the  $\psi$ -part of  $\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} (\mathcal{O}_F^\times / \mathcal{E}_F)$  and the  $\psi$ -part of  $\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F)$  have the same order.

**Mathematics Subject Classification (2010):** 16D10, 11R58 (11R29).

## 1 Introduction.

Let  $K$  be a commutative field,  $A \subseteq K$  be a Dedekind ring and  $V$  be a  $K$ -vector space. In this paper, we associate to every pair of  $A$ -lattices  $R \neq 0$  and  $S$  of  $V$  their  $A$ -index-module

$$[R : S]'_A := \{ \det(u); u \in \text{End}_K(V') \text{ and } u(R) \subseteq S \},$$

where  $V'$  is the  $K$ -subspace of  $V$  generated by  $R$  and  $S$ ,  $V' = \langle R, S \rangle_K$ . As we will see below,  $[R : S]'_A$  is a finitely generated  $A$ -submodule of  $K$ , unless  $\langle R \rangle_K \neq \langle S \rangle_K$  and  $d_R \leq d_S$  (see the notation below). If  $K$  is the fraction field of  $A$ , and  $\langle R \rangle_K = \langle S \rangle_K = V$ , then  $[R : S]'_A$  is equal to the "relative invariant"  $\chi(S, R)$ , defined in [2, §4, n°6, page 63]. Moreover, our  $[R : S]'_A$  recover the notion of the "generalized index", used for instance in [7], [9], [5] and [1]. If  $S \subseteq R$ , then  $[R : S]'_A$  is equal to the  $A$ -Fitting ideal of  $R/S$ ,

$$[R : S]'_A = \text{Fitt}_A(R/S).$$

The index-module have nice properties making it very easily be handled. For example, we consider in section 3 the following situation. Let  $F/k$  be a finite abelian extension of global function fields, with Galois group  $G$ , and degree  $g$ . Let  $\infty$  be a place of  $k$  which splits completely in  $F/k$ . Let  $\mathcal{O}_k$  (resp.  $\mathcal{O}_F$ ) be the ring of functions of  $k$  (resp.  $F$ ), which are regular outside  $\infty$  (resp. the places of  $F$  sitting over  $\infty$ ). Then one may use Stark units to define a subgroup  $\mathcal{E}_F$  of  $\mathcal{O}_F^\times$ , the group of units of  $\mathcal{O}_F$ . The group  $\mathcal{O}_F^\times / \mathcal{E}_F$

---

\*S.Viguié, Laboratoire de mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon cedex, France. e-mail: [stephane.viguié@univ-fcomte.fr](mailto:stephane.viguié@univ-fcomte.fr)

is finite. Since Stark units are related to  $L$ -functions, the order of  $\mathcal{O}_F^\times/\mathcal{E}_F$  is related to the order of the ideal class group  $Cl(\mathcal{O}_F)$  of  $\mathcal{O}_F$ , thanks to the analytic class number formula. We use the notion of index-module to prove that for every nontrivial irreducible rational character  $\psi$  of  $G$ , the  $\psi$ -part of  $\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} (\mathcal{O}_F^\times/\mathcal{E}_F)$  and the  $\psi$ -part of  $\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F)$  have the same order.

## 2 Basic properties of the index-modules.

By an  $A$ -lattice of  $V$ , we mean a finitely generated  $A$ -submodule  $R$  of  $V$ , such that the  $K$ -vector subspace of  $V$  generated by  $R$ , denoted by  $\langle R \rangle_K$  or  $KR$ , has dimension equal to the  $A$ -rank of  $R$ ,  $d_R := \dim_K \langle R \rangle_K = rk_A(R)$ . If  $R \neq 0$ , we know that there exists a fractional ideal  $\mathfrak{m}_R$  of  $A$ , and  $B_R = (b_{R,0}, \dots, b_{R,d_R-1})$  a  $K$ -basis of  $\langle R \rangle_K$ , such that  $R = \mathfrak{m}_R b_{R,0} \oplus \bigoplus_{i=1}^{d_R-1} A b_{R,i}$ . Moreover,  $\mathfrak{m}_R$  can be chosen integral. (See [2, §4, n°10, Proposition 24, page 79].)

In the sequel,  $R \neq 0$ ,  $S$ , and  $T$  are  $A$ -lattices of  $V$ . When  $A$  is implicit, we will simply note  $[R : S]'$ .

**Remark 2.1** For any isomorphism  $u$  of  $V$ , we have  $[u(R) : u(S)]'_A = [R : S]'_A$ .

**Proposition 2.1** We have the following formulas.

$$[R : S]'_A = \begin{cases} 0 & \text{if } d_R > d_S \\ K & \text{if } d_R \leq d_S \text{ and } KS \neq KR \\ \mathfrak{m}_S \mathfrak{m}_R^{-1} \det_{B_S, B_R}(Id_{V'}) & \text{if } V' = KS = KR. \end{cases}$$

*Proof.* The case  $KS \neq KR$  is easy, and left to the reader. In the sequel, we assume  $KS = KR$ . We set  $D = \det_{B_S, B_R}(Id_{V'})$ , and  $n = d_R - 1 = d_S - 1$  for convenience.

Let  $a \in \mathfrak{m}_S \mathfrak{m}_R^{-1}$ , and let  $u_a$  be the unique automorphism of  $V'$  such that  $u_a(b_{R,0}) = a.b_{S,0}$ , and such that for all  $i \in \{1, \dots, n\}$ ,  $u_a(b_{R,i}) = b_{S,i}$ . Clearly  $u_a(R) \subseteq S$ , and

$$\det_{B_R, B_R}(u_a) = \det_{B_R, B_S}(u_a) \det_{B_S, B_R}(Id_{V'}) = aD.$$

From this we deduce  $\mathfrak{m}_S \mathfrak{m}_R^{-1} D \subseteq [R : S]'$ .

Conversely, let  $u$  be an endomorphism of the vector space  $V'$  such that  $u(R) \subseteq S$ . Let  $M := \text{mat}_{B_R, B_S}(u)$  be the matrix of  $u$  with respect to the bases  $B_R$  and  $B_S$ . Then, for all  $a \in \mathfrak{m}_R$ , we have

$$u(ab_{R,0}) \in S \iff \sum_{i=0}^n a M_{i,0} b_{S,i} \in \mathfrak{m}_S b_{S,0} \oplus \bigoplus_{i=1}^n A b_{S,i}.$$

Hence, we have  $M_{0,0} \in \mathfrak{m}_S \mathfrak{m}_R^{-1}$  and  $M_{i,0} \in \mathfrak{m}_R^{-1}$  for all  $i \in \{1, \dots, n\}$ . For all  $j \in \{1, \dots, n\}$ , we have

$$u(b_{R,j}) \in S \iff \sum_{i=0}^n M_{i,j} b_{S,i} \in \mathfrak{m}_S b_{S,0} \oplus \bigoplus_{i=1}^n A b_{S,i}.$$

Hence, we have  $M_{0,j} \in \mathfrak{m}_S$  and  $M_{i,j} \in A$  for all  $i \in \{1, \dots, n\}$ . Let  $\sigma$  be a permutation of  $\{0, \dots, n\}$ . If  $\sigma(0) = 0$ , then for all  $i \in \{1, \dots, n\}$ ,  $\sigma(i) \in \{1, \dots, n\}$ . In this case,  $M_{\sigma(0),0}$  (i.e

$M_{0,0}$ ) belongs to  $\mathfrak{m}_S \mathfrak{m}_R^{-1}$ , and for all  $i \in \{1, \dots, n\}$ ,  $M_{\sigma(i),i} \in A$ . So  $\prod_{i=0}^n M_{\sigma(i),i} \in \mathfrak{m}_S \mathfrak{m}_R^{-1}$ .

If  $\sigma(0) \in \{1, \dots, n\}$ , then  $M_{\sigma(0),0} \in \mathfrak{m}_R^{-1}$ . There is  $l \in \{1, \dots, n\}$  such that  $0 = \sigma(l)$ , and then  $M_{\sigma(l),l}$  (i.e.  $M_{0,l}$ ) belongs to  $\mathfrak{m}_S$ . For all  $j \in \{1, \dots, n\} \setminus \{l\}$ ,  $M_{\sigma(j),j} \in A$ . So  $\prod_{i=0}^n M_{\sigma(i),i} \in \mathfrak{m}_S \mathfrak{m}_R^{-1}$ . This proves that  $\det_{B_R, B_S}(u) \in \mathfrak{m}_S \mathfrak{m}_R^{-1}$ . Since

$$\det(u) = \det_{B_R, B_S}(u) \det_{B_S, B_R}(Id_{V'}) = \det_{B_R, B_S}(u) D,$$

we get  $\det(u) \in \mathfrak{m}_S \mathfrak{m}_R^{-1} D$ . Thus we have verified that  $[R : S]' \subseteq \mathfrak{m}_S \mathfrak{m}_R^{-1} D$ , so that  $[R : S]' = \mathfrak{m}_S \mathfrak{m}_R^{-1} D$ .  $\square$

**Remark 2.2** If  $R = S$ , then we obtain  $[R : R]'_A = A$ .

**Corollary 2.1** The  $A$ -index-module  $[R : S]'_A$  is an  $A$ -module, which is finitely generated unless  $d_R \leq d_S$  and  $KS \neq KR$ .

**Proposition 2.2** Assume that  $KR = KS$ . Then the following properties are equivalent.

- i) The  $A$ -modules  $R$  and  $S$  are isomorphic.
- ii) There is an automorphism  $u$  of  $V'$  such that  $u(R) = S$ .
- iii)  $[R : S]'$  is a cyclic  $A$ -module.

Suppose these properties are satisfied. Then for any automorphism  $v$  of  $V'$  such that  $v(R) = S$ , we have  $[R : S]' = A \cdot \det(v)$ .

*Proof.* i)  $\Rightarrow$  ii) is trivial. If ii) is true, we can choose  $(\mathfrak{m}_R, B_R)$  and  $(\mathfrak{m}_S, B_S)$  such that  $\mathfrak{m}_R = \mathfrak{m}_S$ , and then we deduce iii) from Proposition 2.1. Let  $x \in K$  be such that  $[R : S]' = \mathfrak{m}_S \mathfrak{m}_R^{-1} x$ . If iii) is true, then  $\mathfrak{m}_S \mathfrak{m}_R^{-1}$  is principal, so  $\mathfrak{m}_R$  and  $\mathfrak{m}_S$  are isomorphic as  $A$ -modules, and i) follows.

Finally, assume i), ii), and iii) are true. Let  $v$  be an automorphism of  $V'$  such that  $v(R) = S$ . We choose  $(\mathfrak{m}_R, B_R)$  and  $(\mathfrak{m}_S, B_S)$  such that  $\mathfrak{m}_S = \mathfrak{m}_R$ , and  $b_{S,i} = v(b_{R,i})$  for all  $i \in \{0, \dots, d_R - 1\}$ . Then Proposition 2.1 gives

$$[R : S]' = A \cdot \det_{B_S, B_R}(Id_{V'}) = A \cdot \det_{B_R, B_S}(v) \det_{B_S, B_R}(Id_{V'}) = A \cdot \det(v).$$

$\square$

**Remark 2.3** In [7], W. Sinnott defined a generalized index  $(R : S)$ , for the case  $d_R = d_S = \dim_K(V) < \infty$ , when  $A = \mathbb{Z}$  and  $K = \mathbb{Q}$  or  $K = \mathbb{R}$ , or when  $A = \mathbb{Z}_p$  and  $K = \mathbb{Q}_p$ , with  $p$  a prime number. By definition, if  $u$  is an endomorphism of  $V$  such that  $u(R) = S$ , then:

- $(R : S) = |\det(u)|$  in the case  $A = \mathbb{Z}$  and  $K = \mathbb{Q}$  or  $K = \mathbb{R}$ .
- $(R : S) = p^{v(\det(u))}$  in the case  $A = \mathbb{Z}_p$  and  $K = \mathbb{Q}_p$ , with  $v$  the normalized valuation on  $\mathbb{Q}_p$ .

From Proposition 2.2, we have  $[R : S]'_A = A(R : S)$  in both cases.

**Proposition 2.3** Let  $n = \dim_K(V)$ ,  $(\mathfrak{m}_i)_{i=1}^n$  and  $(\mathfrak{n}_i)_{i=1}^n$  two families of fractional ideals of  $A$ , and  $B = (b_i)_{i=1}^n$  a  $K$ -basis of  $V$ . We set  $M = \bigoplus_{i=1}^n \mathfrak{m}_i b_i$  and  $N = \bigoplus_{i=1}^n \mathfrak{n}_i b_i$ . Then

$$[M : N]' = \prod_{i=1}^n \mathfrak{n}_i \mathfrak{m}_i^{-1}.$$

*Proof.* For all  $i \in \{1, \dots, n\}$ , let  $x_i \in \mathfrak{n}_i \mathfrak{m}_i^{-1}$ . Let  $u$  be the endomorphism of  $V$  such that for all  $i \in \{1, \dots, n\}$ ,  $u(b_i) = x_i b_i$ . Then  $u(M) \subseteq N$ , and  $\det(u) = \prod_{i=1}^n x_i$ . This shows that

$$\prod_{i=1}^n \mathfrak{n}_i \mathfrak{m}_i^{-1} \subseteq [M : N]'$$

Let  $v$  be an endomorphism of  $V$  such that  $v(M) \subseteq N$ . Let  $P = \text{mat}_{B,B}(v)$ . For all  $j \in \{1, \dots, n\}$ , and all  $a \in \mathfrak{m}_j$ , we have

$$v(ab_j) \in S \iff \sum_{i=1}^n P_{i,j} ab_i \in \bigoplus_{i=1}^n \mathfrak{n}_i b_i.$$

So  $P_{i,j} \in \mathfrak{n}_i \mathfrak{m}_j^{-1}$ , for all  $i \in \{1, \dots, n\}$ . For all permutation  $\sigma$  of  $\{1, \dots, n\}$ , we have  $\prod_{i=1}^n P_{\sigma(i),i} \in \prod_{i=1}^n \mathfrak{n}_{\sigma(i)} \mathfrak{m}_i^{-1}$ , and so  $\prod_{i=1}^n P_{\sigma(i),i} \in \prod_{i=1}^n \mathfrak{n}_i \mathfrak{m}_i^{-1}$ . Then  $\det(v) \in \prod_{i=1}^n \mathfrak{n}_i \mathfrak{m}_i^{-1}$ . Thus, we have verified  $[R : S]' \subseteq \prod_{i=1}^n \mathfrak{n}_i \mathfrak{m}_i^{-1}$ .  $\square$

**Proposition 2.4** *Assume  $KR = KS$  or  $KS = KT \neq 0$ . Then*

$$[R : T]' = [R : S]'[S : T]'$$

(This product has to be understood in the following way:  $[R : T]'$  is the sub- $A$ -module of  $K$  generated by all the elements  $xy$ , with  $(x, y) \in [R : S]' \times [S : T]'$ .)

*Proof.* The inclusion  $[R : S]'[S : T]' \subseteq [R : T]'$  is trivial. If  $[R : S]'$  is zero, then  $d_S < d_R$ , and  $KS = KT$ . So  $d_T < d_R$ , and  $[R : T]'$  is zero. A similar argument shows that if  $[S : T]'$  is zero, then  $[R : T]'$  is zero.

If  $[R : S]' \neq 0$  with  $KR \neq KS$ , then  $KS = KT$ ,  $[R : S]' = K$  and  $[S : T]' \neq 0$ . This implies  $K = [R : S]'[S : T]' \subseteq [R : T]'$ , and thus  $[R : T]' = K$ . In the same way, if  $[S : T]' \neq 0$  with  $KS \neq KT$ , then  $[R : T]' = K$ .

To conclude, it suffices to study the case  $KR = KS = KT$ . Then, by Proposition 2.1, we have

$$\begin{aligned} [R : T]' &= \mathfrak{m}_T \mathfrak{m}_R^{-1} \det_{B_T, B_R}(Id_{V'}) \\ &= \mathfrak{m}_S \mathfrak{m}_R^{-1} \det_{B_S, B_R}(Id_{V'}) \mathfrak{m}_T \mathfrak{m}_S^{-1} \det_{B_T, B_S}(Id_{V'}) = [R : S]'[S : T]'. \end{aligned}$$

$\square$

**Corollary 2.2** *Assume that  $KR = KS$ . Then  $[R : S]' = ([S : R]')^{-1}$ , where  $([S : R]')^{-1}$  is the sub- $A$ -module  $\{x \in K; [R : S]'x \subseteq A\}$  of  $K$ .*

**Lemma 2.1** *Let  $V''$  be a nonzero sub- $K$ -vector space of  $V$ . Let  $n = \dim_K(V'')$ ,  $(\mathfrak{m}_i)_{i=1}^n$  and  $(\mathfrak{n}_i)_{i=1}^n$  two families of fractional ideals of  $A$ ,  $B = (b_i)_{i=1}^n$  and  $C = (c_i)_{i=1}^n$  two  $K$ -basis of  $V''$ . We set  $M = \bigoplus_{i=1}^n \mathfrak{m}_i b_i$  and  $N = \bigoplus_{i=1}^n \mathfrak{n}_i c_i$ . Then*

$$[M : N]' = \left( \prod_{i=1}^n \mathfrak{n}_i \mathfrak{m}_i^{-1} \right) \det_{C,B}(Id_{V''}).$$

*Proof.* By Proposition 2.4, we have

$$[M : N]' = [M : \bigoplus_{i=1}^n Ab_i]' [\bigoplus_{i=1}^n Ab_i : \bigoplus_{i=1}^n Ac_i]' [\bigoplus_{i=1}^n Ac_i : N]'$$

From Proposition 2.3, we obtain  $[M : \bigoplus_{i=1}^n Ab_i]' = \prod_{i=1}^n \mathfrak{m}_i^{-1}$ , and  $[\bigoplus_{i=1}^n Ac_i : N]' = \prod_{i=1}^n \mathfrak{n}_i$ . By

Proposition 2.2,  $[\bigoplus_{i=1}^n Ab_i : \bigoplus_{i=1}^n Ac_i]' = A.\det(u)$ , where  $u$  is the unique endomorphism of  $V''$  such that for all  $i \in \{1, \dots, n\}$ ,  $u(b_i) = c_i$ . But  $\det(u) = \det_{C,B}(Id_{V''})$ , and the lemma follows.  $\square$

**Proposition 2.5** (*Direct sums*) Let  $n \in \mathbb{N}^*$ ,  $(R_i)_{i=1}^n$  a family of sub- $A$ -modules of  $R$ , and  $(S_i)_{i=1}^n$  a family of sub- $A$ -modules of  $S$ , such that

$$R = \bigoplus_{i=1}^n R_i \quad \text{and} \quad S = \bigoplus_{i=1}^n S_i.$$

We assume that for all  $i \in \{1, \dots, n\}$ ,  $KS_i \subseteq KR_i \neq 0$ . Then

$$[R : S]' = \prod_{i=1}^n [R_i : S_i]'$$

*Proof.* We have  $KR = \bigoplus_{i=1}^n KR_i$  and  $KS = \bigoplus_{i=1}^n KS_i$ . If there is  $i \in \{1, \dots, n\}$  such that  $KS_i \not\subseteq KR_i$ , then  $KR \neq KS$ ,  $[R : S]' = 0$  and  $[R_i : S_i]' = 0$ , so the result is trivial. Suppose now that for all  $i \in \{1, \dots, n\}$ ,  $KS_i \subseteq KR_i$ . For all  $i \in \{1, \dots, n\}$ , let  $q_i = \dim_K(KR_i)$ ,  $B_i = (b_{i,j})_{j=1}^{q_i}$  and  $C_i = (c_{i,j})_{j=1}^{q_i}$  two  $K$ -basis of  $KR_i$ ,  $(\mathfrak{m}_{i,j})_{j=1}^{q_i}$  and  $(\mathfrak{n}_{i,j})_{j=1}^{q_i}$  two families of fractional ideals of  $A$  such that

$$R_i = \bigoplus_{j=1}^{q_i} \mathfrak{m}_{i,j} b_{i,j} \quad \text{and} \quad S_i = \bigoplus_{j=1}^{q_i} \mathfrak{n}_{i,j} c_{i,j}.$$

We also set  $B = (b_{1,1}, \dots, b_{1,q_1}, \dots, b_{n,1}, \dots, b_{n,q_n})$  and  $C = (c_{1,1}, \dots, c_{1,q_1}, \dots, c_{n,1}, \dots, c_{n,q_n})$ .

These are two  $K$ -basis of  $KR$ . Then we have  $[R_i : S_i]' = \left( \prod_{j=1}^{q_i} \mathfrak{n}_{i,j} \mathfrak{m}_{i,j}^{-1} \right) \det_{C_i, B_i}(Id_{V_i})$ , thanks to Lemma 2.1. But we have the decompositions

$$R = \bigoplus_{i=1}^n \bigoplus_{j=1}^{q_i} \mathfrak{m}_{i,j} b_{i,j} \quad \text{and} \quad S = \bigoplus_{i=1}^n \bigoplus_{j=1}^{q_i} \mathfrak{n}_{i,j} c_{i,j}.$$

Hence, from Lemma 2.1 we deduce

$$\begin{aligned} [R : S]' &= \det_{C,B}(Id_{V'}) \prod_{i=1}^n \prod_{j=1}^{q_i} \mathfrak{n}_{i,j} \mathfrak{m}_{i,j}^{-1} \\ &= \prod_{i=1}^n \left( \prod_{j=1}^{q_i} \mathfrak{n}_{i,j} \mathfrak{m}_{i,j}^{-1} \right) \det_{C_i, B_i}(Id_{V_i}) = \prod_{i=1}^n [R_i : S_i]'. \end{aligned}$$

$\square$

**Proposition 2.6** (*Scalar expansion*) Let  $B$  be a Dedekind ring, embedded in  $K$ , such that  $A$  is a sub-ring of  $B$ . Then  $[BR : BS]'_B = B[R : S]'_A$ . (Where  $B[R : S]'_A$  is the  $B$ -submodule of  $K$  generated by  $[R : S]'_A$ .)

*Proof.* First, notice that  $BR$  is a  $B$ -lattice of  $V$ , thanks to the inequalities

$$rk_A(R) = \dim_K(KR) \leq rk_B(BR) \leq rk_A(R).$$

If  $d_S < d_R$ , then  $[BR : BS]'_B = 0 = [R : S]'_A$ . If  $d_R \leq d_S$ , with  $KS \neq KR$ , then  $[BR : BS]'_B = K = [R : S]'_A$ . Suppose now  $KR = KS$ . By Proposition 2.1, we have

$$[BR : BS]'_B = (B\mathbf{m}_S)(B\mathbf{m}_R)^{-1} \det_{B_S, B_R}(Id_{V'}) = B\mathbf{m}_S \mathbf{m}_R^{-1} \det_{B_S, B_R}(Id_{V'}) = B[R : S]'_A.$$

□

**Proposition 2.7** *Assume  $S \subseteq R$  and  $d_R = d_S$ . If  $A/\mathfrak{a}$  is finite for any nonzero ideal  $\mathfrak{a}$  of  $A$ , then*

$$[A : [R : S]'_A] = [R : S],$$

where  $[A : [R : S]'_A]$  (resp.  $[R : S]$ ) is the group-index of the ideal  $[R : S]'_A$  in  $A$  (resp. of  $S$  in  $R$ ).

*Proof.* If  $A$  is a field then  $R = S$  and  $[R : S]'_A = A$ . In the sequel, we assume  $A$  is not a field, and we consider two cases.

First, suppose  $A$  is a discrete valuation ring. Let  $\pi$  be a uniformizer of  $A$ . We can choose  $\mathbf{m}_R = A$ , and  $B_R$  such that there is  $(m_i)_{i=1}^{d_R} \in \mathbb{N}^{d_R}$ , with  $S = \bigoplus_{i=1}^{d_R} \pi^{m_i} b_{R,i-1}$ . Then  $R/S \simeq \bigoplus_{i=1}^n A/\pi^{m_i} A$ , and  $[R : S]'_A = \pi^{\sum_{i=1}^n m_i} A$ . This implies the proposition if  $A$  is a discrete valuation ring.

Suppose now  $A$  is any Dedekind ring. We have  $R/S \simeq \bigoplus_{\mathfrak{p} \in \text{Spec}^*(A)} A_{\mathfrak{p}} \otimes_A (R/S)$  (where  $\text{Spec}^*(A)$  is the set of nonzero prime ideals of  $A$ ). Since  $A_{\mathfrak{p}}$  is a flat  $A$ -module and a discrete valuation ring, for all  $\mathfrak{p} \in \text{Spec}^*(A)$ , we deduce

$$[R : S] = \prod_{\mathfrak{p} \in \text{Spec}^*(A)} [A_{\mathfrak{p}} R : A_{\mathfrak{p}} S] = \prod_{\mathfrak{p} \in \text{Spec}^*(A)} [A_{\mathfrak{p}} : [A_{\mathfrak{p}} R : A_{\mathfrak{p}} S]_{A_{\mathfrak{p}}}].$$

Taking into account the proposition 2.6, we obtain

$$\prod_{\mathfrak{p} \in \text{Spec}^*(A)} [A_{\mathfrak{p}} : [A_{\mathfrak{p}} R : A_{\mathfrak{p}} S]_{A_{\mathfrak{p}}}] = \prod_{\mathfrak{p} \in \text{Spec}^*(A)} [A_{\mathfrak{p}} : A_{\mathfrak{p}} [R : S]'_A] = [A : [R : S]'_A].$$

□

**Corollary 2.3** *Suppose  $A = \mathcal{S}^{-1}\mathbb{Z}$ , where  $\mathcal{S}$  is a multiplicative part of the ring  $\mathbb{Z}$ , which does not contain 0. If  $S \subseteq R$  and  $d_R = d_S$ , we have  $[R : S]'_A = [R : S]A$ .*

*Proof.* For any nonzero ideal  $\mathfrak{a}$  of  $A$ , we have  $\mathfrak{a} = [A : \mathfrak{a}]A$  and  $A/\mathfrak{a}$  is finite. From Proposition 2.7, we deduce

$$[R : S]'_A = [A : [R : S]'_A] A = [R : S]A.$$

□

For the rest of this section, assume  $K$  is the fraction-field of  $A$ , and  $KR = KS = V$  (in this case,  $[R : S]'$  is a fractional ideal of  $A$ ). Set  $n = \dim_K(V) - 1$ . For any choice of

a  $K$ -basis of  $V$ , we have a canonical isomorphism  $\bigwedge^{n+1} V \simeq K$ . Through this isomorphism, the canonical image of  $R^{n+1}$  in  $\bigwedge^{n+1} V$  is identified to a fractional ideal  $\mathfrak{a}$  of  $A$ . In the same way,  $S^{n+1}$  defines a fractional ideal  $\mathfrak{b}$  of  $A$ . Then we define the relative invariant  $\chi(S, R)$  of  $S$  and  $R$ ,  $\chi(S, R) := \mathfrak{b}\mathfrak{a}^{-1}$ . It does not depend on the choice of the  $K$ -basis of  $V$  (see [2, §4, n°6], for more details and basic properties about the relative invariant). We use a multiplicative notation for the relative invariant, instead of the usual additive one, because it is more adapted to our situation.

**Proposition 2.8**  $\chi(S, R) = [R : S]'$ .

*Proof.* Set  $M = \bigoplus_{i=0}^n Ab_{R,i}$ , and  $N = \bigoplus_{i=0}^n Ab_{S,i}$ .  $b_{R,0} \wedge \cdots \wedge b_{R,n}$  is a basis of  $\bigwedge^{n+1} V$ . The  $A$ -lattice of  $\bigwedge^{n+1} V$ , generated by the canonical image of  $M^{n+1}$ , is  $Ab_{R,0} \wedge \cdots \wedge Ab_{R,n}$ , and the  $A$ -lattice of  $\bigwedge^{n+1} V$ , generated by the canonical image of  $R^{n+1}$  is  $\mathfrak{m}_R b_{R,0} \wedge \cdots \wedge b_{R,n}$ . By definition,  $\chi(M, R) = \mathfrak{m}_R^{-1}$ . In the same way,  $\chi(S, N) = \mathfrak{m}_S$ .

Let  $u$  be the unique endomorphism of  $V$  such that  $u(b_{R,i}) = b_{S,i}$  for all  $i \in \{0, \dots, n\}$ . Then, applying [2, §4, n°6, Proposition 13]:

$$\chi(N, M) = (\det(u)) = (\det_{B_S, B_R}(Id_V) \det_{B_R, B_S}(u)) = (\det_{B_S, B_R}(Id_V))$$

Finally

$$\chi(S, R) = \chi(S, N) \chi(N, M) \chi(M, R) = \mathfrak{m}_S \mathfrak{m}_R^{-1} \det_{B_S, B_R}(Id_V) = [R : S]'$$

□

**Lemma 2.2** Let  $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$  be an exact sequence of finitely generated  $A$ -modules, and assume  $L$  is free.  $L$  and  $N$  are viewed as  $A$ -lattices of  $K \otimes_A L$ . Then

$$Fitt_A(M) = [L : N]'_A.$$

*Proof.* Let  $C = \{c_1, \dots, c_n\}$  be an  $A$ -basis of  $L$ . Let  $\mathcal{M}$  be the set of square  $n \times n$ -matrices  $T := (t_{i,j})$ , with coefficients in  $A$ , such that for all  $j \in \{1, \dots, n\}$ ,  $\left(\sum_{i=1}^n t_{i,j} c_i\right) \in N$ . Then  $Fitt_A(M)$  is the ideal of  $A$  generated by the determinants of the matrices  $T \in \mathcal{M}$ . Let  $T \in \mathcal{M}$ . There is a unique endomorphism  $f$  of  $K \otimes_A L$ , such that for all  $j \in \{1, \dots, n\}$ ,  $f(c_j) = \sum_{i=1}^n t_{i,j} c_i$ . Then  $f(L) \subseteq N$ , and  $\det(f) = \det(T)$ . We deduce the inclusion  $Fitt_A(M) \subseteq [L : N]'_A$ .

Let  $f \in \text{End}_K(K \otimes_A L)$ , such that  $f(L) \subseteq N$ . Let  $T$  be the matrix of  $f$  in the  $K$ -basis  $C$  of  $K \otimes_A L$ . Obviously,  $T \in \mathcal{M}$ , and so  $\det(f) \in Fitt_A(M)$ . Thus we have proved  $[L : N]'_A \subseteq Fitt_A(M)$ . □

**Theorem 2.1** Let  $M$  be a nonzero finitely generated  $A$ -module, torsion-free over  $A$ , and  $N$  an  $A$ -submodule of  $M$ .  $M$  and  $N$  are viewed as  $A$ -lattices of  $K \otimes_A M$ . Then  $Fitt_A(M/N) = [M : N]'_A$ .



*Proof.* Let us write  $M = \mathfrak{m}b_0 \oplus \bigoplus_{i=1}^n Ab_i$ , where  $\mathfrak{m}$  is a nonzero integral ideal of  $A$  and  $(b_0, \dots, b_n)$  is a  $K$ -basis of  $K \otimes_A M$ . We set  $L = \bigoplus_{i=0}^n Ab_i$ . Since  $A$  is a Dedekind ring, the exact sequence

$$0 \rightarrow M/N \rightarrow L/N \rightarrow L/M \rightarrow 0$$

gives  $Fitt_A(L/M)Fitt_A(M/N) = Fitt_A(L/N)$ . By Lemma 2.2 we have

$$[L : M]_A' Fitt_A(M/N) = [L : N]_A'.$$

But  $KM = KL$ . Thus, multiplying by  $[M : L]_A'$  and applying Proposition 2.4 and Remark 2.2, we obtain the desired formula.  $\square$

### 3 The index of Stark units in function fields.

In this section, we apply the notion of index-module to prove Theorem 3.1 below, which gives a weak form of the Gras conjecture in positive characteristic. For this we shall use the following notation.

Let  $k$  be a global function field, with field of constants  $\mathbb{F}_q$ . Let  $\infty$  be a place of  $k$ , of degree  $d$  over  $\mathbb{F}_q$ . Then, we denote by  $k_\infty$  the completion of  $k$  at  $\infty$ . Let  $\mathcal{O}_k$  be the Dedekind ring of functions  $f \in k$  regular outside  $\infty$ . Let us also fix  $F \subseteq k_\infty$ , a finite abelian extension of  $k$  in which  $\infty$  splits completely, with Galois group  $G$  and degree  $g$ .

For any finite abelian extension  $K$  of  $k$ , we denote by  $\mathcal{O}_K$  the integral closure of  $\mathcal{O}_k$  in  $K$ , and by  $\mathcal{O}_K^\times$  the group of units of  $\mathcal{O}_K$ . We denote by  $\mu(K)$  the group of roots of unity in  $K$ , and by  $Cl(\mathcal{O}_K)$  the ideal class group of  $\mathcal{O}_K$ .

#### 3.1 Stark units in function fields.

If  $\mathfrak{m}$  is a nonzero ideal of  $\mathcal{O}_k$  then we denote by  $H_\mathfrak{m} \subseteq k_\infty$  the maximal abelian extension of  $k$  contained in  $k_\infty$ , such that the conductor of  $H_\mathfrak{m}/k$  divides  $\mathfrak{m}$ . In particular,  $\infty$  splits completely in  $H_\mathfrak{m}/k$ . The function field version of the abelian conjectures of Stark, proved by P. Deligne in [8] by using étale cohomology or by D. Hayes in [3] by using Drinfel'd modules, claims that, for any proper nonzero ideal  $\mathfrak{m}$  of  $\mathcal{O}_k$ , there exists an element  $\varepsilon_\mathfrak{m} \in H_\mathfrak{m}$ , unique up to roots of unity, such that

- (i) The extension  $H_\mathfrak{m}(\varepsilon_\mathfrak{m}^{1/w_\infty})/k$  is abelian, where  $w_\infty := q^d - 1$ .
- (ii) If  $\mathfrak{m}$  is divisible by two prime ideals then  $\varepsilon_\mathfrak{m}$  is a unit of  $\mathcal{O}_{H_\mathfrak{m}}$ . If  $\mathfrak{m} = \mathfrak{q}^e$ , where  $\mathfrak{q}$  is a prime ideal of  $\mathcal{O}_k$  and  $e$  is a positive integer, then

$$\varepsilon_\mathfrak{m} \mathcal{O}_{H_\mathfrak{m}} = (\mathfrak{q})_\mathfrak{m}^{\frac{w_\infty}{w_k}},$$

where  $w_k := q - 1$  and  $(\mathfrak{q})_\mathfrak{m}$  is the product of the prime ideals of  $\mathcal{O}_{H_\mathfrak{m}}$  which divide  $\mathfrak{q}$ .

(iii) We have

$$L_\mathfrak{m}(0, \chi) = \frac{1}{w_\infty} \sum_{\sigma \in \text{Gal}(H_\mathfrak{m}/k)} \chi(\sigma) v_\infty(\varepsilon_\mathfrak{m}^\sigma) \quad (3.1)$$

for all complex irreducible characters of  $\text{Gal}(H_\mathfrak{m}/k)$ , where  $v_\infty$  is the normalized valuation of  $k_\infty$ .

Let us recall that  $s \mapsto L_\mathfrak{m}(s, \chi)$  is the  $L$ -function associated to  $\chi$ , defined for the complex numbers  $s$  such that  $\text{Re}(s) > 1$  by the Euler product

$$L_\mathfrak{m}(s, \chi) = \prod_{v \nmid \mathfrak{m}} (1 - \chi(\sigma_v) N(v)^{-s})^{-1},$$



where  $v$  describes the set of places of  $k$  not dividing  $\mathfrak{m}$ . For such a place,  $\sigma_v$  and  $N(v)$  are the Frobenius automorphism of  $H_{\mathfrak{m}}/k$  and the order of the residue field at  $v$  respectively. Let us remark that  $\sigma_{\infty} = 1$  and  $N(\infty) = q^d$ .

For any finite abelian extension  $L$  of  $k$  we denote by  $\mathcal{J}_L \subseteq \mathbb{Z}[\text{Gal}(L/k)]$  the annihilator of  $\mu(L)$ . The description of  $\mathcal{J}_L$  given in [3, Lemma 2.5] and the property (i) of  $\varepsilon_{\mathfrak{m}}$  implies that for any  $\eta \in \mathcal{J}_{H_{\mathfrak{m}}}$  there exists  $\varepsilon_{\mathfrak{m}}(\eta) \in H_{\mathfrak{m}}$  such that

$$\varepsilon_{\mathfrak{m}}(\eta)^{w_{\infty}} = \varepsilon_{\mathfrak{m}}^{\eta}.$$

**Definition 3.1** *Let  $\mathcal{P}_F$  be the subgroup of  $F^*$  generated by  $\mu(F)$  and by all the norms*

$$N_{H_{\mathfrak{m}}/H_{\mathfrak{m}} \cap F}(\varepsilon_{\mathfrak{m}}(\eta)),$$

where  $\mathfrak{m}$  is any nonzero proper ideal of  $\mathcal{O}_k$ , and  $\eta \in \mathcal{J}_{H_{\mathfrak{m}}}$ . By definition, the group of Stark units is

$$\mathcal{E}_F = \mathcal{P}_F \cap \mathcal{O}_F^{\times}.$$

**Remark 3.1** *The index  $[\mathcal{O}_F^{\times} : \mathcal{E}_F]$  is finite. This will be proved in the next subsection. In [4], H. Oukhaba succeeded in computing this index in case  $F \subseteq H_{(1)}$ . He obtained the following formula*

$$[\mathcal{O}_F^{\times} : \mathcal{E}_F] = \frac{h(\mathcal{O}_F)}{[H_{(1)} : F]},$$

where  $h(\mathcal{O}_F)$  is the ideal class number of  $\mathcal{O}_F$ .

Let  $S$  be a set of places of  $k$ , which contains  $\infty$ . In [6], C. Popescu defined a group  $\mathcal{E}_S$  of  $S$ -units of  $F$  by using Rubin-Stark units. If  $S = \{\infty\}$ , he proved that for any prime number  $p$ ,  $p \nmid g$ , and every nontrivial irreducible  $p$ -adic character  $\psi$  of  $G$ , the Gras conjecture is verified,

$$\#(\mathbb{Z}_p \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times} / \mathcal{E}_S))_{\psi} = \#(\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\psi},$$

where the subscript  $\psi$  means we take the  $\psi$ -parts. See [6, Theorem 3.10].

In the sequel, we use index-modules to prove a weak form of the analogous statement for the group  $\mathcal{E}_F$  (see Theorem 3.1), i.e for rational characters. It can be shown that  $\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} \mathcal{E}_F$  is included in  $\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} \mathcal{E}_S$ . From Theorem 3.1, it follows that this inclusion is an equality, so that the full Gras conjecture is also true for  $\mathcal{E}_F$ .

### 3.2 An index formula for Stark units.

Let  $\ell_F : F^{\times} \rightarrow \mathbb{Z}[G]$  be the  $G$ -equivariant map defined by

$$\ell_F(x) = \sum_{\sigma \in G} v_{\infty}(x^{\sigma}) \sigma^{-1}.$$

Let  $\mu_g$  be the group of  $g$ -th roots of unity in the field of complex numbers. Let  $\mathcal{O}$  be the integral closure of the principal ring  $\mathbb{Z}_{\langle g \rangle} := \mathbb{Z}[g^{-1}]$  in  $\mathbb{Q}(\mu_g)$ . Let us denote by  $\widehat{G}$  the group of complex irreducible characters of  $G$ . Then, for every  $\chi \in \widehat{G}$  the idempotent  $e_{\chi} := \frac{1}{g} \sum_{\sigma \in G} \chi(\sigma) \sigma^{-1}$  belongs to  $\mathcal{O}[G]$ . Moreover, if  $\zeta \in \mu_g$  is such that  $\zeta \neq 1$ , then

$1 - \zeta \in \mathcal{O}^{\times}$ , thanks to the formula  $g = \prod_{\substack{\zeta \in \mu_g \\ \zeta \neq 1}} (1 - \zeta)$ .

**Definition 3.2** Let  $\Omega$  be the  $\mathbb{Z}[G]$ -submodule of  $F^*$  generated by  $\mu(F)$  and the elements of the form  $N_{H_{\mathfrak{m}}/F \cap H_{\mathfrak{m}}}(\varepsilon_{\mathfrak{m}})$ , where  $\mathfrak{m}$  is any nonzero proper ideal of  $\mathcal{O}_k$ .

**Proposition 3.1** Let  $\chi \in \widehat{G}$  be such that  $\chi \neq 1$ . Let  $\chi_{pr}$  be the character of  $\text{Gal}(H_{\mathfrak{f}_{\chi}}/k)$  deduced from  $\chi$ , where  $\mathfrak{f}_{\chi}$  is the conductor of the fixed field  $F_{\chi}$  of  $\text{Ker}(\chi)$ . Then

$$\mathcal{O}\ell_F(\Omega)e_{\chi} = \mathcal{O}w_{\infty}L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr})e_{\chi}, \quad (3.2)$$

where  $\mathcal{O}\ell_F(\Omega) \subseteq \mathcal{O}[G]$  is the  $\mathcal{O}$ -module generated by  $\ell_F(\Omega)$ .

*Proof.* The equality (3.2) is a direct consequence of the property (iii) of Stark units stated above. We take our inspiration from the computation made in [5]. Let  $\mathfrak{m}$  be a nonzero proper ideal of  $\mathcal{O}_k$  and let  $\varepsilon_{F, \mathfrak{m}} := N_{H_{\mathfrak{m}}/F \cap H_{\mathfrak{m}}}(\varepsilon_{\mathfrak{m}})$ . If  $\chi$  is not trivial on  $\text{Gal}(F/F \cap H_{\mathfrak{m}})$ , then

$$\ell_F(\varepsilon_{F, \mathfrak{m}})e_{\chi} = 0.$$

But, if  $\chi$  is trivial on  $\text{Gal}(F/F \cap H_{\mathfrak{m}})$  then  $F_{\chi} \subseteq H_{\mathfrak{m}}$ ,  $\mathfrak{f}_{\chi} | \mathfrak{m}$  and

$$\ell_F(\varepsilon_{F, \mathfrak{m}})e_{\chi} = [F : F \cap H_{\mathfrak{m}}] \left( \sum_{\sigma \in \text{Gal}(H_{\mathfrak{m}}/k)} \overline{\chi'(\sigma)} v_{\infty}(\varepsilon_{F, \mathfrak{m}}^{\sigma}) \right) e_{\chi},$$

where  $\chi'$  is the complex character of  $\text{Gal}(H_{\mathfrak{m}}/k)$  deduced from  $\chi$ . Therefore the equality (3.1) gives

$$\ell_F(\varepsilon_{F, \mathfrak{m}})e_{\chi} = w_{\infty}[F : F \cap H_{\mathfrak{m}}] L_{\mathfrak{m}}(0, \bar{\chi}')e_{\chi}.$$

Now, the relation

$$L_{\mathfrak{m}}(0, \bar{\chi}) = \prod_{\substack{v | \mathfrak{m} \\ v \nmid \mathfrak{f}_{\chi}}} (1 - \bar{\chi}_{pr}(\sigma_v)) L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr})$$

clearly shows that  $\ell_F(\varepsilon_{F, \mathfrak{m}})e_{\chi} \in \mathcal{O}w_{\infty}L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr})e_{\chi}$ . Conversely, the hypothesis that  $\chi$  is not trivial implies that there exists some prime ideal  $\mathfrak{p}$  of  $\mathcal{O}_k$  such that  $\chi_{pr}(\sigma_{\mathfrak{p}}) \neq 1$ . Thus, if we put  $\mathfrak{m} := \mathfrak{p}\mathfrak{f}_{\chi}$  then

$$\ell_F(\varepsilon_{F, \mathfrak{m}})e_{\chi} = w_{\infty}[F : F \cap H_{\mathfrak{m}}] (1 - \bar{\chi}_{pr}(\sigma_{\mathfrak{p}})) L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr})e_{\chi}.$$

But, since  $[F : F \cap H_{\mathfrak{m}}]$  and  $(1 - \bar{\chi}_{pr}(\sigma_{\mathfrak{p}}))$  are in  $\mathcal{O}^{\times}$  we obtain  $w_{\infty}L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr})e_{\chi} \in \mathcal{O}\ell_F(\Omega)e_{\chi}$ . The proposition is now proved.  $\square$

If  $\psi$  is an irreducible character of  $G$ , then we denote by  $\mathcal{X}_{\psi}$  the set of  $\chi \in \widehat{G}$  such that  $\chi | \psi$ . Let  $M$  be a  $\mathbb{Z}_{\langle g \rangle}[G]$ -module. Thus we put  $M_{\psi} = e_{\psi}M$ , where  $e_{\psi} = \sum_{\chi \in \mathcal{X}_{\psi}} e_{\chi}$ . If  $M$  is

an  $\mathcal{O}[G]$ -module and  $\chi \in \widehat{G}$  then we put  $M_{\chi} := e_{\chi}M$ .

**Corollary 3.1** Let  $\psi \neq 1$  be an irreducible rational character of  $G$ . Then  $(\mathcal{O}\ell_F(\Omega))_{\psi}$  and  $(\mathcal{O}\mathcal{J}_F\ell_F(\Omega))_{\psi}$  are  $\mathcal{O}$ -lattices of the  $\mathbb{C}$ -vector space  $\mathbb{C}[G]$ . Moreover,

$$\left[ (\mathcal{O}\ell_F(\Omega))_{\psi} : (\mathcal{O}\mathcal{J}_F\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' = \mathcal{O} \# (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mu(F))_{\psi}.$$

*Proof.* On one hand, we have the decomposition

$$(\mathcal{O}\ell_F(\Omega))_\psi = \bigoplus_{\chi \in \mathcal{X}_\psi} (\mathcal{O}\ell_F(\Omega))_\chi = \bigoplus_{\chi \in \mathcal{X}_\psi} (\mathcal{O}w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{pr}))_\chi, \quad (3.3)$$

the last equality being an application of Proposition 3.1. On the other hand, since  $\psi \neq 1$  the primitive character  $\chi_{pr}$  is nontrivial for all  $\chi \in \mathcal{X}_\psi$ . As a consequence,  $L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{pr}) \neq 0$ . This implies that  $(\mathcal{O}\ell_F(\Omega))_\psi$  is a free  $\mathcal{O}$ -module of rank  $\#(\mathcal{X}_\psi) = \dim(\psi)$ , and hence, it is an  $\mathcal{O}$ -lattice of  $\mathbb{C}[G]$ . Similar arguments may be used to verify that  $(\mathcal{O}\mathcal{J}_F\ell_F(\Omega))_\psi$  is an  $\mathcal{O}$ -lattice of  $\mathbb{C}[G]$ . Furthermore, using Proposition 2.5, Proposition 3.1 and Remark 2.1, we obtain

$$\begin{aligned} \left[ (\mathcal{O}\ell_F(\Omega))_\psi : (\mathcal{O}\mathcal{J}_F\ell_F(\Omega))_\psi \right]_{\mathcal{O}}' &= \prod_{\chi \in \mathcal{X}_\psi} \left[ (\mathcal{O}\ell_F(\Omega))_\chi : (\mathcal{O}\mathcal{J}_F\ell_F(\Omega))_\chi \right]_{\mathcal{O}}' \\ &= \prod_{\chi \in \mathcal{X}_\psi} \left[ (\mathcal{O}w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{pr}))_\chi : (\mathcal{O}\mathcal{J}_F w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{pr}))_\chi \right]_{\mathcal{O}}' \\ &= \prod_{\chi \in \mathcal{X}_\psi} \left[ \mathcal{O}[G]_\chi : (\mathcal{O}\mathcal{J}_F)_\chi \right]_{\mathcal{O}}'. \end{aligned} \quad (3.4)$$

By Proposition 2.5 and Proposition 2.6, we have

$$\prod_{\chi \in \mathcal{X}_\psi} \left[ \mathcal{O}[G]_\chi : (\mathcal{O}\mathcal{J}_F)_\chi \right]_{\mathcal{O}}' = [\mathcal{O}[G]_\psi : (\mathcal{O}\mathcal{J}_F)_\psi]_{\mathcal{O}}' = \mathcal{O} \left[ \mathbb{Z}_{\langle g \rangle}[G]_\psi : (\mathbb{Z}_{\langle g \rangle}\mathcal{J}_F)_\psi \right]_{\mathbb{Z}_{\langle g \rangle}}' \quad (3.5)$$

By Corollary 2.3, we have

$$\left[ \mathbb{Z}_{\langle g \rangle}[G]_\psi : (\mathbb{Z}_{\langle g \rangle}\mathcal{J}_F)_\psi \right]_{\mathbb{Z}_{\langle g \rangle}}' = \mathbb{Z}_{\langle g \rangle} \left[ \mathbb{Z}_{\langle g \rangle}[G]_\psi : (\mathbb{Z}_{\langle g \rangle}\mathcal{J}_F)_\psi \right]. \quad (3.6)$$

From (3.4), (3.5), (3.6) and the definition of  $\mathcal{J}_F$ , we obtain

$$\left[ (\mathcal{O}\ell_F(\Omega))_\psi : (\mathcal{O}\mathcal{J}_F\ell_F(\Omega))_\psi \right]_{\mathcal{O}}' = \mathcal{O} \left[ \mathbb{Z}_{\langle g \rangle}[G]_\psi : (\mathbb{Z}_{\langle g \rangle}\mathcal{J}_F)_\psi \right] = \mathcal{O} \# (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mu(F))_\psi.$$

Whence the corollary.  $\square$

To go further we need some preliminary remarks.

**Remark 3.2** *Let  $H$  be a sub-group of  $G$ . Let  $M$  and  $N$  be two  $G$ -modules, and  $\psi : M \rightarrow N$  be a  $G$ -equivariant map. If  $\text{Coker}(\Psi) := N/\text{Im}(\Psi)$  is annihilated by  $\#(H)$  then we derive from  $\Psi$  a surjective map*

$$\Psi_{\mathcal{O}} : \mathcal{O} \otimes_{\mathbb{Z}} M \twoheadrightarrow \mathcal{O} \otimes_{\mathbb{Z}} N.$$

*Let us assume, in addition, that  $\text{Ker}(\Psi)$  is annihilated by  $\Sigma\sigma$ ,  $\sigma \in H$ . Then, for every  $\chi \in \hat{G}$  trivial on  $H$ , the restriction of  $\Psi_{\mathcal{O}}$  gives an isomorphism*

$$(\mathcal{O} \otimes_{\mathbb{Z}} M)_\chi \simeq (\mathcal{O} \otimes_{\mathbb{Z}} N)_\chi.$$

As a particular case, for any subextension  $K/k$  of  $F/k$  and  $H = \text{Gal}(F/K)$ , we shall consider the norm maps  $Cl(\mathcal{O}_F) \rightarrow Cl(\mathcal{O}_K)$ ,  $\mu(F) \rightarrow \mu(K)$ , and also the map

$$\mathbb{Z}[G]_0/\ell_F(\mathcal{O}_F^\times) \rightarrow \mathbb{Z}[\text{Gal}(K/k)]_0/\ell_K(\mathcal{O}_K^\times),$$

deduced from the natural map  $\mathbb{Z}[G] \rightarrow \mathbb{Z}[\text{Gal}(K/k)]$ , where  $\mathbb{Z}[G]_0$  (resp.  $\mathbb{Z}[\text{Gal}(K/k)]_0$ ) is the augmentation ideal of  $\mathbb{Z}[G]$  (resp.  $\mathbb{Z}[\text{Gal}(K/k)]$ ).

**Remark 3.3** For any commutative rings  $A \subseteq B$  and any finitely generated  $A$ -module  $M$ ,  $Fitt_B(B \otimes_A M) = B \cdot Fitt_A(M)$ . In particular, let  $M$  be a finite  $\mathbb{Z}[G]$ -module, let  $\psi$  be an irreducible rational character of  $G$ . Then

$$\mathcal{O} \# \left( (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} M)_{\psi} \right) = \prod_{\chi \in \mathcal{X}_{\psi}} Fitt_{\mathcal{O}} \left( (\mathcal{O} \otimes_{\mathbb{Z}} M)_{\chi} \right).$$

**Proposition 3.2** Let  $\psi \neq 1$  be an irreducible rational character of  $G$ . Then

$$\left[ (\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\psi} : (\mathcal{O} \ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' = \mathcal{O} \frac{w_{\infty}^{\dim(\psi)} \# (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_{\psi}}{\# (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mu(F))_{\psi}}, \quad (3.7)$$

where  $(\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\psi}$  and  $(\mathcal{O} \ell_F(\Omega))_{\psi}$  are viewed as  $\mathcal{O}$ -lattices of the  $\mathbb{C}$ -vector space  $\mathbb{C}[G]$ .

*Proof.* Obviously,  $F_{\chi}$  does not depend on the choice of  $\chi \in \mathcal{X}_{\psi}$ . We set  $F_{\psi} := F_{\chi}$ , for  $\chi \in \mathcal{X}_{\psi}$ . Let  $\Xi_{\psi}$  be the set of  $\chi \in \widehat{G}$  such that  $\text{Ker } \chi$  strictly contains  $\text{Gal}(F/F_{\psi})$ . For any  $I \subseteq \Xi_{\psi}$ , we define

$$F_I := \begin{cases} F_{\psi} & \text{if } I = \emptyset. \\ \bigcap_{\chi \in I} F_{\chi} & \text{if } I \neq \emptyset. \end{cases}$$

Using Proposition 2.5 and the formula (3.2), we have

$$[\mathcal{O}[G]_{\psi} : \mathcal{O} \ell_F(\Omega)_{\psi}]_{\mathcal{O}}' = \prod_{\chi \in \mathcal{X}_{\psi}} [\mathcal{O}[G]_{\chi} : (\mathcal{O} \ell_F(\Omega))_{\chi}]_{\mathcal{O}}' = \prod_{\chi \in \mathcal{X}_{\psi}} [\mathcal{O}[G]_{\chi} : (\mathcal{O} w_{\infty} L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr}))_{\chi}]_{\mathcal{O}}'.$$

By Proposition 2.2, and since  $\#(\mathcal{X}_{\psi}) = \dim(\psi)$ ,

$$[\mathcal{O}[G]_{\psi} : \mathcal{O} \ell_F(\Omega)_{\psi}]_{\mathcal{O}}' = \mathcal{O} w_{\infty}^{\dim(\psi)} \prod_{\chi \in \mathcal{X}_{\psi}} L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr}). \quad (3.8)$$

For  $I \subseteq \Xi_{\psi}$  fixed we denote by  $\Xi_I$  the set of  $\chi \in \widehat{G}$  such that  $F_{\chi} \subseteq F_I$ . The inclusion-exclusion principle and the analytic class number formula give

$$\prod_{\chi \in \mathcal{X}_{\psi}} L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr}) = \prod_{I \subseteq \Xi_{\psi}} \left( res(\zeta_k, 0) \prod_{\substack{\chi \in \Xi_I \\ \chi \neq 1}} L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr}) \right)^{(-1)^{\#(I)}} = \prod_{I \subseteq \Xi_{\psi}} (res(\zeta_{F_I}, 0))^{(-1)^{\#(I)}}, \quad (3.9)$$

where  $\zeta_{F_I}$  (resp.  $\zeta_k$ ) is the Dedekind zeta function of  $F_I$  (resp.  $k$ ) and  $res(\zeta_{F_I}, 0)$  (resp.  $res(\zeta_k, 0)$ ) is the residue of  $\zeta_{F_I}$  (resp.  $\zeta_k$ ) at 0. Let  $\mathbb{Z}[\text{Gal}(F_I/k)]_0$  be the augmentation ideal of  $\mathbb{Z}[\text{Gal}(F_I/k)]$ . Then, it can be shown that

$$res(\zeta_{F_I}, 0) = -\frac{h(\mathcal{O}_{F_I}) \mathcal{R}(\mathcal{O}_{F_I})}{w_{F_I} \ln(q^d)}, \quad (3.10)$$

where  $\mathcal{R}(\mathcal{O}_{F_I}) := [\mathbb{Z}[\text{Gal}(F_I/k)]_0 : \ell_{F_I}(\mathcal{O}_{F_I}^{\times})]$ . Since  $\mathbb{Z}h(\mathcal{O}_{F_I}) = Fitt_{\mathbb{Z}}(Cl(\mathcal{O}_{F_I}))$  we have

$$\begin{aligned} \mathcal{O}h(\mathcal{O}_{F_I}) = \mathcal{O}Fitt_{\mathbb{Z}}(Cl(\mathcal{O}_{F_I})) &= Fitt_{\mathcal{O}}(\mathcal{O} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_{F_I})) \\ &= \prod_{\chi \in \Xi_I} Fitt_{\mathcal{O}}((\mathcal{O} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_{F_I}))_{\chi}) \\ &= \prod_{\chi \in \Xi_I} Fitt_{\mathcal{O}}((\mathcal{O} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_{\chi}). \end{aligned}$$

The last equality is an application of Remark 3.2. In the same manner we have

$$\mathcal{O}w_{F_I} = \prod_{\chi \in \Xi_I} \text{Fitt}_{\mathcal{O}} \left( (\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_{\chi} \right)$$

$$\text{and } \mathcal{O}\mathcal{R}(\mathcal{O}_{F_I}) = \prod_{\chi \in \Xi_I} \text{Fitt}_{\mathcal{O}} \left( (\mathcal{O} \otimes_{\mathbb{Z}} (\mathbb{Z}[G]_0/\ell_F(\mathcal{O}_F^{\times}))_{\chi} \right).$$

For any  $\chi \in \widehat{G}$ , let us set  $h_{\chi} := \text{Fitt}_{\mathcal{O}} \left( (\mathcal{O} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_{\chi} \right)$ ,  $w_{\chi} := \text{Fitt}_{\mathcal{O}} \left( (\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_{\chi} \right)$ , and  $\mathcal{R}_{\chi} := \text{Fitt}_{\mathcal{O}} \left( (\mathcal{O} \otimes_{\mathbb{Z}} (\mathbb{Z}[G]_0/\ell_F(\mathcal{O}_F^{\times}))_{\chi} \right)$ . Combining (3.9) and (3.10), and applying the inclusion-exclusion principle a second time we obtain

$$\begin{aligned} \mathcal{O} \prod_{\chi \in \mathcal{X}_{\psi}} L_{\mathfrak{f}_{\chi}}(0, \bar{\chi}_{pr}) &= \mathcal{O} \prod_{I \subseteq \Xi_{\psi}} \left( \frac{h(\mathcal{O}_{F_I}) \mathcal{R}(\mathcal{O}_{F_I})}{-w_{F_I} \ln(q^d)} \right)^{(-1)^{\#(I)}} = \prod_{I \subseteq \Xi_{\psi}} \left( \prod_{\chi \in \Xi_I} h_{\chi} \mathcal{R}_{\chi} w_{\chi}^{-1} \right)^{(-1)^{\#(I)}} \\ &= \prod_{\chi \in \mathcal{X}_{\psi}} h_{\chi} \mathcal{R}_{\chi} w_{\chi}^{-1}. \end{aligned} \quad (3.11)$$

By (3.8), (3.11), Remark 3.3, Corollary 2.3, and Proposition 2.6, we have

$$\left[ \mathcal{O}[G]_{\psi} : (\mathcal{O}\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' = \frac{w_{\infty}^{\dim(\psi)} \#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_{\psi}}{\#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mu(F))_{\psi}} \left[ \mathcal{O}[G]_{\psi} : (\mathcal{O}\ell(\mathcal{O}_F^{\times}))_{\psi} \right]_{\mathcal{O}}'.$$

Multiplying by  $\left[ (\mathcal{O}\ell(\mathcal{O}_F^{\times}))_{\psi} : \mathcal{O}[G]_{\psi} \right]_{\mathcal{O}}'$ , and applying Proposition 2.4, we have

$$\left[ (\mathcal{O}\ell_F(\mathcal{O}_F^{\times}))_{\psi} : (\mathcal{O}\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' = \mathcal{O} \frac{w_{\infty}^{\dim(\psi)} \#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_{\psi}}{\#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mu(F))_{\psi}}.$$

□

**Corollary 3.2** *Let  $\psi \neq 1$  be an irreducible rational character of  $G$ . Then*

$$\left[ (\mathcal{O}\ell_F(\mathcal{O}_F^{\times}))_{\psi} : (\mathcal{O}w_{\infty}^{-1}\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' = \mathcal{O} \frac{\#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_{\psi}}{\#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mu(F))_{\psi}}.$$

*Proof.* We have the equality

$$\begin{aligned} \left[ (\mathcal{O}\ell_F(\mathcal{O}_F^{\times}))_{\psi} : (\mathcal{O}w_{\infty}^{-1}\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' &= \\ \left[ (\mathcal{O}\ell_F(\mathcal{O}_F^{\times}))_{\psi} : (\mathcal{O}\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' \left[ (\mathcal{O}\ell_F(\Omega))_{\psi} : (\mathcal{O}w_{\infty}^{-1}\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}', \end{aligned}$$

thanks to Proposition 2.4. By Proposition 2.2, (iii), we have

$$\left[ (\mathcal{O}\ell_F(\Omega))_{\psi} : (\mathcal{O}w_{\infty}^{-1}\ell_F(\Omega))_{\psi} \right]_{\mathcal{O}}' = w_{\infty}^{-\dim(\psi)}.$$

We conclude by using Proposition 2.4 and the above Proposition 3.2. □

**Theorem 3.1** *Let  $\psi \neq 1$  be an irreducible rational character of  $G$ . We have*

$$\left[ (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{O}_F^\times)_\psi : (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{E}_F)_\psi \right] = \# (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_\psi.$$

*Proof.* We deduce from the identity  $\mathcal{O}_F^\times \cap \text{Ker}(\ell_F) = \mu(F)$  that the factor  $G$ -modules  $\mathcal{O}_F^\times / \mathcal{E}_F$  and  $\ell_F(\mathcal{O}_F^\times) / \ell_F(\mathcal{E}_F)$  are isomorphic. In particular,

$$\left[ (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{O}_F^\times)_\psi : (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{E}_F)_\psi \right] = \left[ (\mathbb{Z}_{\langle g \rangle} \ell_F(\mathcal{O}_F^\times))_\psi : (\mathbb{Z}_{\langle g \rangle} \ell_F(\mathcal{E}_F))_\psi \right]. \quad (3.12)$$

Let us also remark that, by the property (ii) of Stark units,  $\mathcal{P}_F^{\sigma-1} \subseteq \mathcal{E}_F$  for all  $\sigma \in G$ . Thus, since  $\psi \neq 1$  and  $g \in \mathcal{O}^\times$  we obtain  $(\mathcal{O} \ell_F(\mathcal{E}_F))_\psi = (\mathcal{O} \ell_F(\mathcal{P}_F))_\psi$ . But  $\ell_F(\mathcal{P}_F)$  and  $\ell_F(\Omega)$  are related by the equality  $w_\infty \ell_F(\mathcal{P}_F) = \mathcal{J}_F \ell_F(\Omega)$ , so that

$$(\mathcal{O} \ell_F(\mathcal{E}_F))_\psi = (\mathcal{O} w_\infty^{-1} \mathcal{J}_F \ell_F(\Omega))_\psi. \quad (3.13)$$

Therefore, Corollary 2.3, Proposition 2.6 and the formula (3.13) give

$$\begin{aligned} \mathcal{O} \left[ (\mathbb{Z}_{\langle g \rangle} \ell_F(\mathcal{O}_F^\times))_\psi : (\mathbb{Z}_{\langle g \rangle} \ell_F(\mathcal{E}_F))_\psi \right] &= \mathcal{O} \left[ (\mathbb{Z}_{\langle g \rangle} \ell_F(\mathcal{O}_F^\times))_\psi : (\mathbb{Z}_{\langle g \rangle} \ell_F(\mathcal{E}_F))_\psi \right]_{\mathbb{Z}_{\langle g \rangle}}' \\ &= \left[ (\mathcal{O} \ell_F(\mathcal{O}_F^\times))_\psi : (\mathcal{O} w_\infty^{-1} \mathcal{J}_F \ell_F(\Omega))_\psi \right]_{\mathcal{O}}'. \end{aligned}$$

By Proposition 2.4, this last  $\mathcal{O}$ -index-module is the product of the two index-modules already computed in Corollary 3.1 (see also Remark 2.1) and Corollary 3.2. Thus we obtain

$$\mathcal{O} \left[ (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{O}_F^\times)_\psi : (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{E}_F)_\psi \right] = \mathcal{O} \# \left( (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} Cl(\mathcal{O}_F))_\psi \right).$$

Since the integers we are comparing are prime to  $g$ , the theorem follows.  $\square$

**Acknowledgement.** I wish to express all my gratitude to Hassan Oukhaba, who introduced me to this topic.

## References

- [1] Jean-Robert Belliard and T. Nguyen-Quang-Do, *On modified circular units and annihilation of real classes*, Nagoya Mathematical Journal **177** (2005), 77-115.
- [2] N. Bourbaki, *Algèbre commutative, chapitre 7, Diviseurs*, Hermann, 1965.
- [3] David R. Hayes, *Stickelberger elements in function fields*, Compositio Mathematica **55** (1985), 209-239.
- [4] Hassan Oukhaba, *Groups of Elliptic Units in Global Function Fields*, The Arithmetic of Function Fields (David Goss, David R. Hayes, and Michael I. Rosen, eds.), Walter de Gruyter, (1992), 87-102.
- [5] Hassan Oukhaba, *Unités elliptiques, indice et  $\mathbb{Z}_p$ -extensions*, Annales mathématiques Blaise Pascal **16** (2009), 165-188.
- [6] Cristian D. Popescu, *Gras-Type Conjectures for Function Fields*, Compositio Mathematica **118** (1999), 263-290.

- [7] W. Sinnott, *On the Stickelberger Ideal and the Circular Units of an Abelian Field*, Inventiones mathematicae **62** (1980), 181-234.
- [8] John Tate, *Les Conjectures de Stark sur les Fonctions L d'Artin en  $s = 0$ .*, Progress in Mathematics, vol. 47, Birkhäuser, (1984).
- [9] Linsheng Yin, *Index-class number formulas over global function fields*, Compositio Mathematica **109** (1997), 49-66.